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THE NONLINEAR RESPONSE OF A FAMILY OF SIMPLE THREE-DIMENSIONAL TRUSSES

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Abstract—In this paper we demonstrate, through a very simple three dimensional truss, a degenerated case of bifurcation-infinite bifurcated paths. In addition we show its implication on numerical simulation by the finite element method. The effect of imperfection on the truss is also discussed, where isolated solutions are found. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Recently much attention has been focused on the study of nonlinear dynamic systems, which show unusual richness in their behaviour, although the systems themselves may be extremely simple (Seydel, 1988, Nashie, 1990, Troger and Steindl, 1991, Shi, 1991). In this paper, we will study a simple nonlinear 3D truss which shows some interesting features that do not exist in linear range. These features may be used as benchmarks to test the solution procedures of a finite element code (Pecknold *et al.*, 1985, Duxbury *et al.*, 1989). Here we restrict ourselves to the phenomenon of infinite bifurcation and isolated solution paths. The main objective is to construct a simple structure model as concept demonstrator and test case for numerical algorithms. During the course of finishing this paper, it has come to the author's notice some similar examples on infinite bifurcation (Ikeda *et al.*, 1988 and Choong, 1994). However these examples are different from the present one in that they require an infinite number of bars (Ikeda *et al.*, 1988) or springs (Choong, 1994). Further imperfection and issues related to numerical simulation have not been addressed. Haughton (1979) and Shilkrut (1991) have both demonstrated isolated solution paths through their elaborate models, but neither of them has discussed aspects of numerical simulation. It will be shown later that a standard finite element solution procedure will not be able to find the isolated solution, which could be extremely dangerous, because potential stability loss due to dynamic buckling may have been ignored.

We consider the following 3D truss made of n identical bars with one end joined together and the other ends evenly distributed on a unit circle in x-y plane (see Fig. 1). For such a simple structure with only three degrees of freedom, we will prove first that it has an infinite number of secondary equilibrium paths, i.e. a degenerated case of bifurcation. Next the effect of this degenerated bifurcation on numerical simulation will be investigated. In the end we will show how the truss behaves when imperfection is present.

2. THE PERFECT TRUSS

Since the truss is "quasi-axisymmetric", we will work in a cylindrical co-ordinate system with the displacement field expressed as:

$$u = r\cos\theta$$
$$v = r\sin\theta$$
$$w = w \tag{1}$$

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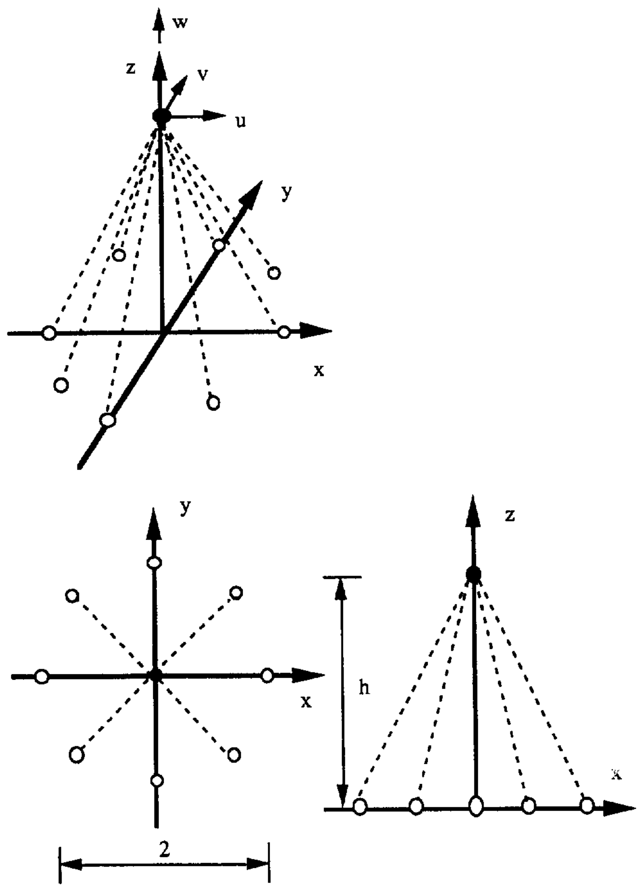

Fig. 1. A simple three dimensional truss.

where u, v and w are the three displacements at the top of the truss, while r and θ are the radius from and rotation around the z axis.

For easy formulation, we will employ the Green strain. The Green's strain for each bar can be easily worked out as:

$$\varepsilon_i = 0.5[(r\cos\theta - \cos\theta_i)^2 + (r\sin\theta - \sin\theta_i)^2 + (h+w)^2 - (1+h^2)]/(1+h^2)$$
$$= 0.5(\varepsilon_i' - l_0^2)/l_0^2 \quad (2)$$

where h is the original height of the truss, l_0 the original length, $\theta_i = i2\pi/n$ and

$$\varepsilon_i' = (r^2 - 2r\cos\theta\cos\theta_i - 2r\sin\theta\sin\theta_i + 1) + (h+w)^2. \quad (3)$$

In order to find the equilibrium equation, we need the Total Potential Energy Π of the truss:

$$\Pi = \Sigma 1/2 EAl_0\varepsilon_i^2 - \lambda w. \quad (4)$$

Here, Σ means summation of i from 1 to n; E is the Young's Modulus, A the cross section area and λ represents the vertical loading at the top of the truss.

By differentiating the potential energy given in eqn (4) with respect to the three degrees of freedom, r, θ and w, we can establish the equilibrium equations:

(a) $\partial\Pi/\partial r = \Sigma EAl_0\varepsilon_i\,\partial\varepsilon_i/\partial r$

$$= \Sigma(\varepsilon_i' - l_0^2)[2r - 2\cos\theta\cos\theta_i - 2\sin\theta\sin\theta_i]$$

$$= 2r\Sigma(\varepsilon_i' - l_0^2) - 2\cos\theta(\Sigma\varepsilon_i\cos\theta_i - l_0^2\Sigma\cos\theta_i) - 2\sin\theta(\Sigma\varepsilon_i\sin\theta_i - l_0^2\Sigma\sin\theta_i)$$

$$= 2rn[r^2 + 1 + (h+w)^2 - l_0^2] + 4r\cos^2\theta\Sigma\cos^2\theta_i + 4r\sin^2\theta\Sigma\sin^2\theta_i + 2r\sin 2\theta\Sigma\sin 2\theta_i$$

$$= 2rn[r^2 + 1 + (h+w)^2 - l_0^2] + 4r(\sin^2\theta + \cos^2\theta)\Sigma\sin^2\theta_i$$

$$= 2rn[r^2 + 1 + (h+w)^2 - l_0^2] + 2r\Sigma(1 - \cos 2\theta_i)$$

$$= 2rn[r^2 + 2 + (h+w)^2 - l_0^2]. \tag{5}$$

Notice in the above we have used relations: $0.25EA/l_0^3 = 1$; $\Sigma\cos\theta_i = 0$; $\Sigma\sin\theta_i = 0$; $\Sigma\sin 2\theta_i = 0$; $\Sigma\cos 2\theta_i = 0$; $\Sigma\cos^2\theta_i = \Sigma\sin^2\theta_i$. For proof see Appendix A.

(b)

$$\partial\Pi/\partial\theta = \Sigma EAl_0\varepsilon_i\,\partial\varepsilon_i/\partial\theta$$

$$= \Sigma(\varepsilon_i' - l_0^2)(2r\sin\theta\cos\theta_i - 2r\cos\theta\sin\theta_i)$$

$$= 2r[\sin\theta(\Sigma\varepsilon_i\cos\theta_i - l_0^2\Sigma\cos\theta_i) - \cos\theta(\Sigma\varepsilon_i\sin\theta_i - l_0^2\Sigma\sin\theta_i)]$$

$$= 2r[2r\sin\theta\cos\theta\Sigma(-\cos^2\theta_i + \sin^2\theta_i) - 2r\sin^2\theta\Sigma\sin\theta_i\cos\theta_i + 2r\cos^2\theta\Sigma\sin\theta_i\cos\theta_i]$$

$$= -2r^2[\sin 2\theta\Sigma\cos 2\theta_i - (\sin^2\theta - \cos^2\theta)\Sigma\sin 2\theta_i]$$

$$\equiv 0. \tag{6}$$

This identity has great bearings on the structure response, which will be further discussed later.

(c)

$$\partial\Pi/\partial w = \Sigma EAl_0\varepsilon_i\,\partial\varepsilon_i/\partial w - \lambda$$

$$= \Sigma(\varepsilon_i' - l_0^2)2(h+w) - \lambda$$

$$= 2n(h+w)[r^2 + 1 + (h+w)^2 - l_0^2] - \lambda. \tag{7}$$

From eqns (5)–(7) we can tell that there is a primary solution:

$$\lambda = 2n(h+w)[1 + (h+w)^2 - l_0^2] = 2n(w^2 + 2wh)(h+w)$$
$$r = 0. \tag{8}$$

On the other hand there is also another secondary solution or bifurcated solution:

$$\lambda = -2n(h+w)$$
$$r^2 + 2 + (h+w)^2 - l_0^2 = 0 \tag{9}$$

which can be written as:

$$\lambda = -2n(h+w)$$
$$r^2 + (h+w)^2 = h^2 - 1. \tag{10}$$

Equation (10) will give us the critical load/displacement at the bifurcation point ($r = 0$):

$$\lambda = -2n(h+w_c)$$
$$w_c = -h \pm (h^2 - 1)^{1/2}. \tag{11}$$

One important thing about the equilibrium equations is that they are independent of θ and the secondary paths form a spherical surface with a radius of $(h^2-1)^{1/2}$ and centred at the origin. To put it in another way, we have an infinite number of secondary paths!!! This may seem to be a little dubious at first sight. But after a careful second thought, it may not look so unusual. Because the truss is in fact a close analogy to an Euler strut with a polygon cross section, which gives equal second moment of inertia, hence the same bending stiffness and buckling load in any transverse directions (only initial buckling is considered). Apparently such a strut can have an infinite number of bifurcated paths due to its axial symmetry.

3. THE STIFFNESS MATRIX AND THE FINITE ELEMENT SIMULATION

In this section, we will show the inherent difficulty in finite element analysis of the truss due to the infinite bifurcation. To this end, we need to derive the tangent stiffness matrix in analytical form, by which we can show that the tangent stiffness matrix is always singular.

By differentiating eqns (5), (6) and (7) with respect to r, θ and w, we can get the tangent stiffness matrix:

$$\mathbf{K}_\theta = \begin{bmatrix} 2n[r^2+2+(h+w)^2-l_0^2]+4r^2n & 0 & 4rn(h+w) \\ 0 & 0 & 0 \\ 4rn(h+w) & 0 & 2n[r^2+1+3(h+w)^2-l_0^2] \end{bmatrix}. \quad (12)$$

which, for the primary path ($r=0$), can be simplified as:

$$\mathbf{K}_\theta = \begin{bmatrix} 2n[2+(h+w)^2-l_0^2] & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2n[1+3(h+w)^2-l_0^2] \end{bmatrix}. \quad (13)$$

For the secondary path, $r^2+2+(h+w)^2-l_0^2=0$, the tangent stiffness matrix becomes:

$$\mathbf{K}_\theta = \begin{bmatrix} 4r^2n & 0 & 4rn(h+w) \\ 0 & 0 & 0 \\ 4rn(h+w) & 0 & 2n[2(h+w)^2-1] \end{bmatrix}. \quad (14)$$

Now we want to find the stiffness matrix in Cartesian co-ordinates system, because it is employed in finite element formulation. To achieve this, we need the transformation matrix $\mathbf{T}$ between the two co-ordinate systems.

From definition we know that:

$$\begin{aligned} r &= (x^2+y^2)^{1/2} \\ \tan\theta &= x/y \\ z &= z \end{aligned} \quad (15)$$

which will lead to the relation:

$$\begin{aligned} \delta r &= (x\delta x+y\delta y)/(x^2+y^2)^{1/2} = \delta x\cos\theta+\delta y\sin\theta \\ \delta\theta &= (1+tg^2\theta)^{-1}(y\delta x-x\delta y)y^{-2} = (y\delta x-x\delta y)/(x^2+y^2) = (\delta x\sin\theta-\delta y\cos\theta)/r \\ \delta z &= \delta z. \end{aligned} \quad (16)$$

As a result the virtual displacement in the cartesian coordinates system $\delta\mathbf{P}_x$, and its counterpart in the cylindrical system $\delta\mathbf{P}_\theta$ are related as:

$$\delta \mathbf{P}_\theta = \begin{pmatrix} \delta r \\ \delta\theta \\ \delta z \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta/r & \cos\theta/r & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} = \mathbf{T}\delta\mathbf{P}_x. \tag{17}$$

We can also find the relation between the forces in the two systems:

$$\delta \mathbf{F}_\theta = \begin{pmatrix} F_r \\ M_\theta \\ F_z \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \mathbf{T}\mathbf{F}_x \tag{18}$$

Substitute eqns (17) and (18) into the following equation:

$$\mathbf{K}_\theta \delta \mathbf{P}_\theta = \delta \mathbf{F}_\theta \tag{19}$$

we can get:

$$\mathbf{K}_\theta \mathbf{T} \delta \mathbf{P}_x = \mathbf{T}' \delta \mathbf{F}_x \tag{20}$$

or,

$$\mathbf{T}'^{-1} \mathbf{K}_\theta \mathbf{T} \delta \mathbf{P}_x = \delta \mathbf{F}_x \tag{21}$$

so,

$$\mathbf{K}_x = \mathbf{T}'^{-1} \mathbf{K}_\theta \mathbf{T}. \tag{22}$$

Applying the above relations to $\mathbf{K}_x$ on the secondary path, we have:

$$\mathbf{K}_x = \begin{bmatrix} 4r^2 n\cos^2\theta & 4r^2 n\cos\theta\sin\theta & 4rn(h+w)\cos\theta \\ 4r^2 n\cos\theta\sin\theta & 4r^2 n\sin^2\theta & 4rn(h+w)\sin\theta \\ 4rn(h+w)\cos\theta & 4rn(h+w)\sin\theta & 2n[2(h+w)^2-1] \end{bmatrix}. \tag{23}$$

One obvious thing about the above matrix is that the first and second rows are linearly dependent on each other. So it is singular, which should not be a surprise at all, as it is singular in the cylindrical system. However in numerical analysis, because of finite machine precision, the round-off errors will result in a nearly singular matrix. Numerical experiment shows that ill conditioning can be handled in some cases, while in others it causes oscillations. To illustrate this, we run a finite element analysis of a four bar truss ($h = 2$). After bracketing the bifurcation point, we do a branch switching by simple eignmode injection (Wagner, 1988). The secondary paths formed by the u and v displacements in x and y directions, as shown in Fig. 2. By using different perturbation patterns, we get either straight

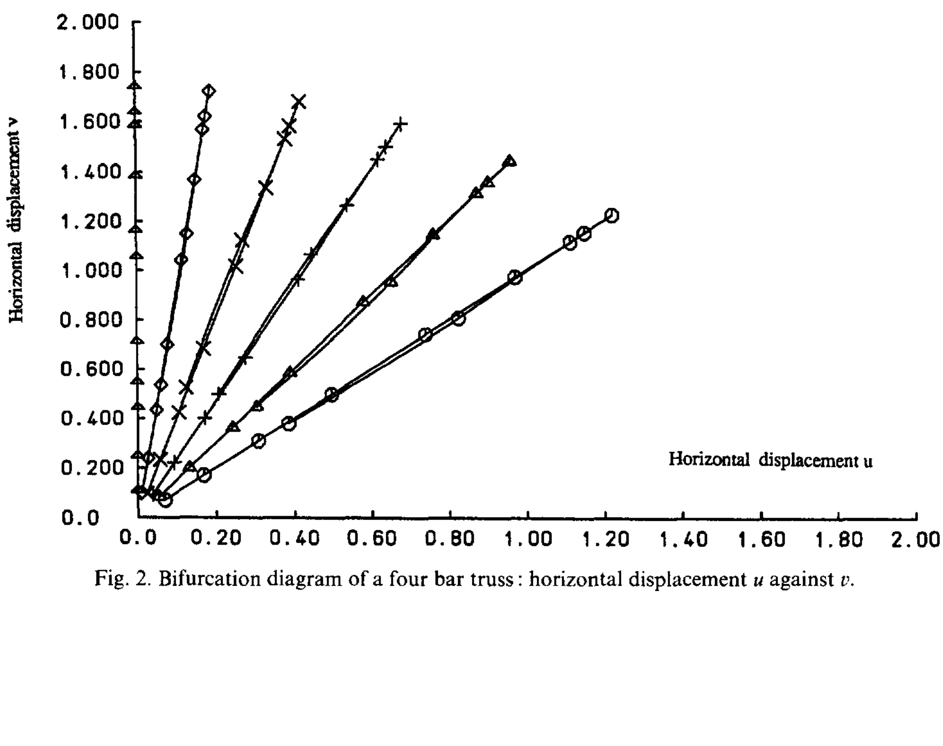

Fig. 2. Bifurcation diagram of a four bar truss: horizontal displacement u against v.

or zigzagged bifurcated paths. This confirms firstly that there are an infinite number of paths; secondly that certain paths are more numerically stable (straight lines) than the other (zigzagged lines). It can also be proved numerically that all the secondary paths are of the same shape (results are not shown here).

4. UNSYMMETRIC LOADING

Now we consider the case that the truss is subject to loading:

$$F_x = \kappa\lambda\cos\theta_0$$
$$F_y = \kappa\lambda\sin\theta_0$$
$$F_z = \lambda \tag{24}$$

where κ denotes constant load imperfection.

Under such a loading the total potential energy is:

$$\begin{aligned}\Pi &= \Sigma 1/2EAl_0\varepsilon_i^2 - \kappa\lambda\cos\theta_0 r\cos\theta - \kappa\lambda\sin\theta_0 r\sin\theta - \lambda w\\ &= \Sigma 1/2EAl_0\varepsilon_i^2 - \kappa\lambda r\cos(\theta-\theta_0) - \lambda w.\end{aligned} \tag{25}$$

We can proceed in the same way to get the equilibrium equations:

$$\begin{aligned}\text{(a)}\quad \partial\Pi/\partial r &= \Sigma EAl_0\varepsilon_i\,\partial\varepsilon_i/\partial r\\ &= 2rn[r^2+2+(h+w)^2-l_0^2]-\kappa\lambda\cos(\theta-\theta_0)\end{aligned} \tag{26}$$

$$\text{(b)}\quad \partial\Pi/\partial\theta = \kappa\lambda r\sin(\theta-\theta_0) \tag{27}$$

$$\text{(c)}\quad \partial\Pi/\partial w = 2n(h+w)[r^2+1+(h+w)^2-l_0^2]-\lambda = 0. \tag{28}$$

To satisfy eqn (27), one needs to have:

$$r = 0, \quad \text{or} \quad \sin(\theta-\theta_0) = 0. \tag{29}$$

Below we will consider each case separately.

(a) $r = 0$. We have, from eqn (26), $\theta-\theta_0 = \pm\pi/2$ and from eqn (28), $\lambda = 2n(w^2+2wh)(h+w)$. However when $r = 0$, $\theta-\theta_0 = \pm\pi/2$ is meaningless. So this is not a solution. Physically it is impossible to have a deformation path with $r = 0$ and a non-zero side force κ.

(b) $\sin(\theta-\theta_0) = 0$ or $\theta-\theta_0 = 0; \pi$. In this case we have, from eqn (26),

$$2rn[r^2+2+(h+w)^2-l_0^2] = \pm\kappa\lambda. \tag{30}$$

Working with eqn (28) we can derive:

$$2n(h+w)[r^2+2wh+w^2] = \lambda. \tag{31}$$

From eqn (30) and eqn (31), we can easily solve for λ and w in terms of r (see Appendix

C). Apparently, the structure response (see Fig. 3, for $n = 4$, $h = 2$, $\kappa = 0.1$) is an unfolding of the circle (see eqn 11) related to the perfect truss. Instead of infinite paths, there is only one single curve in the plane defined by $\theta - \theta_0 = 0$; π. However, as expected, the response is independent of the direction, θ_0, of the load imperfection, which has been confirmed by numerical experiments.

5. GEOMETRIC IMPERFECTION

We consider the same 3D truss with geometric imperfection i.e. the apex of the truss has now been displaced from the original symmetrical position $(0, 0, h)$ to $(r_0 \cos\theta_0, r_0 \sin\theta_0, h)$.

We will still work in a cylindrical coordinate system with the displacement field:

$$
\begin{aligned}
u &= r\cos\theta \\
v &= r\sin\theta \\
w &= w.
\end{aligned} \tag{32}
$$

The variables in eqn (32) are the same as before, but they are measured from $(r_0 \cos\theta_0, r_0 \sin\theta_0, h)$, see Fig. 4.

Because of the imperfection, the bar lengths and strains are different:

$$
\begin{aligned}
\text{old length } l_{oi}^2 &= (\cos\theta_i - r_0\cos\theta_0)^2 + (\sin\theta_i - r_0\sin\theta_0)^2 + h^2 \\
&= 1 + h^2 + r_0^2 - 2r_0\cos(\theta_i - \theta_0) \\
&\approx 1 + h^2 - 2r_0\cos(\theta_i - \theta_0).
\end{aligned} \tag{33}
$$

In the last step we have ignored the second order term r_0^2. In the sequel r_0^2 and higher terms will be omitted;

$$
\begin{aligned}
\text{New length } l_{ni}^2 &= (r\cos\theta + r_0\cos\theta_0 - \cos\theta_i)^2 + (r\sin\theta + r_0\sin\theta_0 - \sin\theta_i)^2 + (w+h)^2 \\
&\approx (w+h)^2 + 1 + r^2 - 2r\cos(\theta_i - \theta) + 2rr_0\cos(\theta_0 - \theta) - 2r_0\cos(\theta_i - \theta_0).
\end{aligned} \tag{34}
$$

Again r_0^2 have been omitted;

$$
\begin{aligned}
\text{strains}: \varepsilon_i &= 0.5(l_{ni}^2 - l_{oi}^2)/l_{oi}^2 \\
&\approx 0.5[(w^2 + 2wh + r^2) - 2r\cos(\theta_i - \theta) + 2rr_0\cos(\theta - \theta_0)]/l_{oi}^2 \\
&= 0.5[\langle 1\rangle - 2r\langle 2\rangle + 2rr_0\langle 4\rangle]/l_{oi}^2 \\
&= 0.5\varepsilon_{ti}/l_{oi}^2
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
\text{where } \langle 1\rangle &= (w^2 + 2wh + r^2) \\
\langle 2\rangle &= \cos(\theta_i - \theta) \\
\langle 3\rangle &= \cos(\theta_i - \theta_0)/(h^2 + 1) \\
\langle 4\rangle &= \cos(\theta - \theta_0).
\end{aligned} \tag{36}
$$

For later use we now derive an approximation of $1/l_{oi}^3$:

$$
1/l_{oi}^3 = [1 + h^2 + r_0^2 - 2r_0\cos(\theta_i - \theta_0)]^{-3/2}. \tag{37}
$$

If we ignore the second order term, we have:

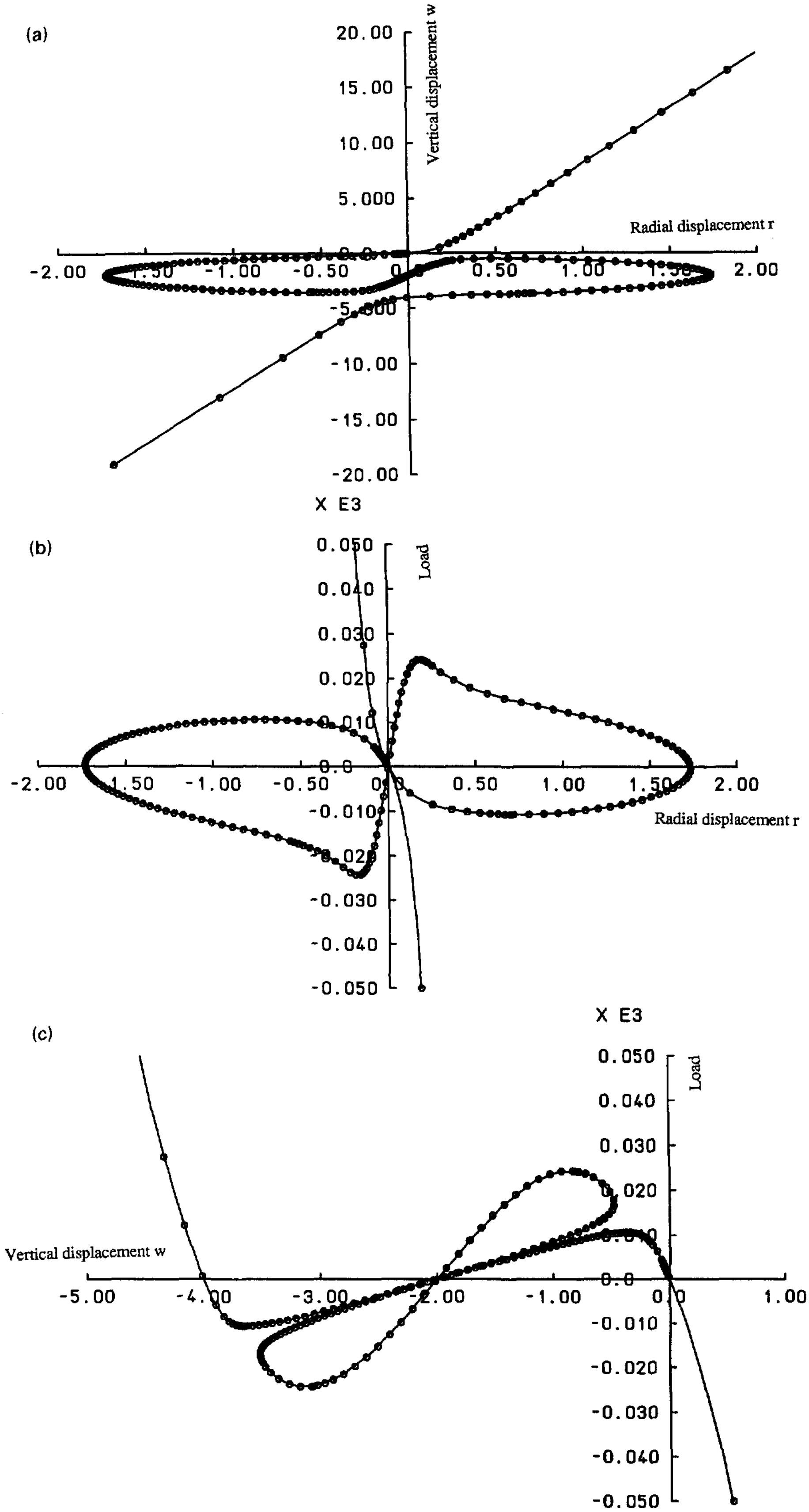

Fig. 3. Structural response with load imperfection: 'o' analytical; '——' finite element. (a) Vertical displacement w against radial displacement r; (b) load against radial displacement; (c) load against vertical displacement.

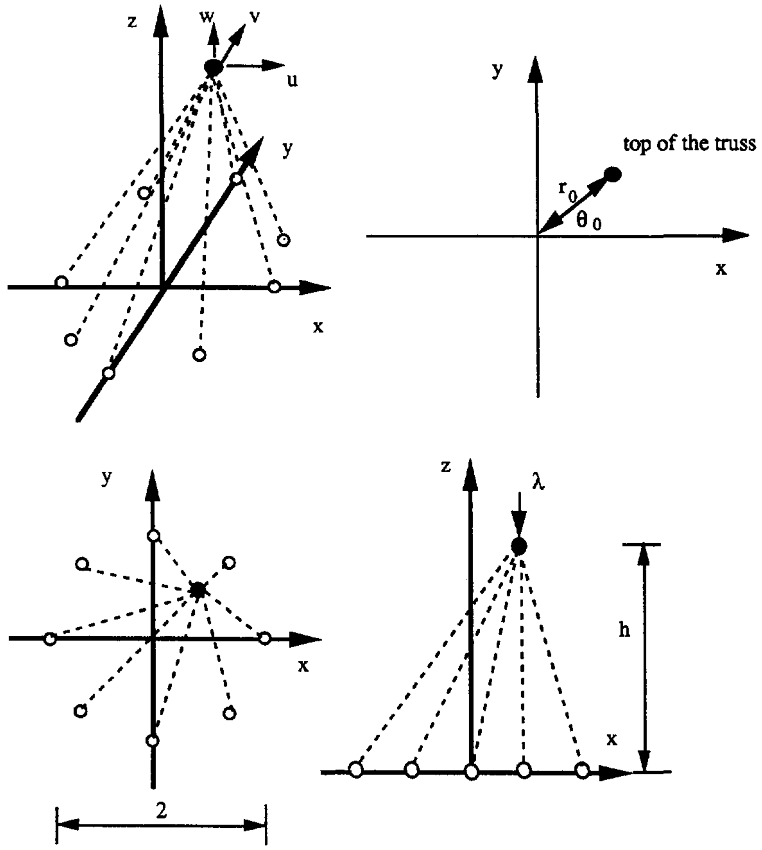

Fig. 4. A simple three dimensional truss with initial geometric imperfection.

$$
\begin{aligned}
1/l_{oi}^3 &= [1+h^2-2r_0\cos(\theta_i-\theta_0)]^{-3/2} \\
&= (1+h^2)^{-3/2}[1-2r_0\cos(\theta_i-\theta_0)/(1+h^2)]^{-3/2}
\end{aligned} \tag{38}
$$

which can be further approximated as:

$$
1/l_{oi}^3 = (1+h^2)^{-3/2}[1+3r_0\cos(\theta_i-\theta_0)/(1+h^2)]. \tag{39}
$$

The Total Potential Energy Π of the truss is:

$$
\Pi = \Sigma 1/2EAl_{0i}\varepsilon_i^2 - \lambda w \tag{40}
$$

from which we can get the equilibrium equations:

$$
\begin{aligned}
\text{(a)}\quad \partial\Pi/\partial r &= \Sigma 0.25EAl_{0i}^{-3}\varepsilon_{ti}\,\partial\varepsilon_{ti}/\partial r \\
&= \Sigma 0.25EA(h^2+1)^{-1.5}[1+3r_0\cos(\theta_i-\theta_0)/(h^2+1)][\langle 1\rangle - 2r\langle 2\rangle + 2rr_0\langle 4\rangle] \\
&\quad \times [2r-2\langle 2\rangle + 2r_0\langle 4\rangle] \\
&= \Sigma[1+3r_0\langle 3\rangle][\langle 1\rangle - 2r\langle 2\rangle + 2rr_0\langle 4\rangle][2r-2\langle 2\rangle + 2r_0\langle 4\rangle] \\
&= \Sigma[1+3r_0\langle 3\rangle][\langle 1\rangle - 2r\langle 2\rangle + 2rr_0\langle 4\rangle][2r-2\langle 2\rangle + 2r_0\langle 4\rangle]
\end{aligned} \tag{41}
$$

in the above we have used eqn (37) and assumed $0.25EA/(h^2+1)^{3/2} = 1$.

Expanding eqn (40), we get:

$$\begin{aligned}\partial\Pi/\partial r = \Sigma[&\langle 1\rangle 2r + 3r_0\langle 3\rangle\langle 1\rangle 2r - 2r\langle 2\rangle 2r - 3r_0\langle 3\rangle 2r\langle 2\rangle 2r \\ &+ 2rr_0\langle 4\rangle 2r + 3r_0\langle 3\rangle 2rr_0\langle 4\rangle 2r - \langle 1\rangle 2\langle 2\rangle - 3r_0\langle 3\rangle\langle 1\rangle 2\langle 2\rangle \\ &+ 2r\langle 2\rangle 2\langle 2\rangle + 3r_0\langle 3\rangle 2r\langle 2\rangle 2\langle 2\rangle - 2rr_0\langle 4\rangle 2\langle 2\rangle - 3r_0\langle 3\rangle 2rr_0\langle 4\rangle 2\langle 2\rangle \\ &+ \langle 1\rangle 2r_0\langle 4\rangle + 3r_0\langle 3\rangle\langle 1\rangle 2r_0\langle 4\rangle - 2r\langle 2\rangle 2r_0\langle 4\rangle - 3r_0\langle 3\rangle 2r\langle 2\rangle 2r_0\langle 4\rangle \\ &+ 2rr_0\langle 4\rangle 2r_0\langle 4\rangle + 3r_0\langle 3\rangle 2rr_0\langle 4\rangle 2r_0\langle 4\rangle]. \end{aligned} \tag{42}$$

Neglecting second order and higher terms of r_0 in the above equation, we arrive at

$$\begin{aligned}\partial\Pi/\partial r = \Sigma[&\langle 1\rangle 2r + 3r_0\langle 3\rangle\langle 1\rangle 2r - 2r\langle 2\rangle 2r - 3r_0\langle 3\rangle 2r\langle 2\rangle 2r \\ &+ 2rr_0\langle 4\rangle 2r - \langle 1\rangle 2\langle 2\rangle - 3r_0\langle 3\rangle\langle 1\rangle 2\langle 2\rangle \\ &+ 2r\langle 2\rangle 2\langle 2\rangle + 3r_0\langle 3\rangle 2r\langle 2\rangle 2\langle 2\rangle - 2rr_0\langle 4\rangle 2\langle 2\rangle \\ &+ \langle 1\rangle 2r_0\langle 4\rangle - 2r\langle 2\rangle 2r_0\langle 4\rangle]. \end{aligned} \tag{43}$$

Based the fact that $\Sigma\langle 2\rangle = \Sigma\langle 3\rangle = 0$ (see Appendix A) and $\Sigma\langle 1\rangle\langle 2\rangle = \Sigma\langle 1\rangle\langle 3\rangle = \Sigma\langle 2\rangle\langle 4\rangle = 0$ (since $\Sigma\langle 1\rangle\langle 2\rangle = \langle 1\rangle\Sigma\langle 2\rangle$, $\Sigma\langle 2\rangle\langle 4\rangle = \langle 4\rangle\Sigma\langle 2\rangle$ and $\Sigma\langle 1\rangle\langle 3\rangle = \langle 1\rangle\Sigma\langle 3\rangle$), we have,

$$\begin{aligned}\partial\Pi/\partial r &= \Sigma\{\langle 1\rangle 2r - [12r^2 r_0 + 6r_0\langle 1\rangle]\langle 2\rangle\langle 3\rangle + 4r\langle 2\rangle\langle 2\rangle + 12rr_0\langle 2\rangle\langle 2\rangle\langle 3\rangle \\ &\quad + (4r^2 r_0 + \langle 1\rangle 2r_0)\langle 4\rangle\} \\ &= 2nr\langle 1\rangle - [12r^2 r_0/(h^2+1) + 6r_0\langle 1\rangle/(h^2+1)]n/2\cos(\theta-\theta_0) + 4rn/2 \\ &\quad + n(4r^2 r_0 + \langle 1\rangle 2r_0)\cos(\theta-\theta_0) \\ &= n(h^2+1)^{-1}\{2r(w^2+2wh+1+r^2)(h^2+1) - r_0[6r^2+3(w^2+2wh+r^2) \\ &\quad - (h^2+1)(6r^2+2w^2+4wh)]\cos(\theta-\theta_0)\}. \end{aligned} \tag{44}$$

Notice in the above we have used relations: $\Sigma\langle 2\rangle\langle 2\rangle = n/2$; $\Sigma\langle 2\rangle\langle 3\rangle = n/2\cos(\theta-\theta_0)$; $\Sigma\langle 2\rangle\langle 3\rangle\langle 3\rangle = 0$. For proof see Appendix B.

(b) $$\begin{aligned}\partial\Pi/\partial\theta &= \Sigma 0.25EAl_{0i}^{-3}\varepsilon_{ti}\,\partial\varepsilon_{ti}/\partial\theta \\ &= \Sigma 0.25EA(h^2+1)^{-1.5}[1+3r_0\cos(\theta_i-\theta_0)/(h^2+1)][\langle 1\rangle - 2r\langle 2\rangle + 2rr_0\langle 4\rangle] \\ &\quad \times[-2r\sin(\theta_i-\theta) - 2rr_0\sin(\theta-\theta_0)] \\ &= \Sigma[1+3r_0\langle 3\rangle][\langle 1\rangle - 2r\langle 2\rangle + 2rr_0\langle 4\rangle][-2r\langle 5\rangle - 2rr_0\langle 6\rangle] \\ &= \Sigma[-\langle 1\rangle 2r\langle 5\rangle + 2r\langle 2\rangle 2r\langle 5\rangle - 2rr_0\langle 4\rangle 2r\langle 5\rangle - 3r_0\langle 3\rangle\langle 1\rangle 2r\langle 5\rangle \\ &\quad + 3r_0\langle 3\rangle 2r\langle 2\rangle 2r\langle 5\rangle - 3r_0\langle 3\rangle 2rr_0\langle 4\rangle 2r\langle 5\rangle - \langle 1\rangle 2rr_0\langle 6\rangle \\ &\quad - 3r_0\langle 3\rangle\langle 1\rangle 2rr_0\langle 6\rangle + 2r\langle 2\rangle 2rr_0\langle 6\rangle + 3r_0\langle 3\rangle 2r\langle 2\rangle 2rr_0\langle 6\rangle \\ &\quad - 2rr_0\langle 4\rangle 2rr_0\langle 6\rangle - 3r_0\langle 3\rangle 2rr_0\langle 4\rangle 2rr_0\langle 6\rangle] \end{aligned} \tag{45}$$

where $\langle 5\rangle = \sin(\theta_i-\theta)$, $\langle 6\rangle = \sin(\theta-\theta_0)$.

Again we neglect r_0^2 and higher order terms:

$$\begin{aligned}\partial\Pi/\partial\theta &= \Sigma[-\langle 1\rangle 2r\langle 5\rangle + 2r\langle 2\rangle 2r\langle 5\rangle - 2rr_0\langle 4\rangle 2r\langle 5\rangle \\ &\quad - 3r_0\langle 3\rangle\langle 1\rangle 2r\langle 5\rangle + 3r_0\langle 3\rangle 2r\langle 2\rangle 2r\langle 5\rangle - \langle 1\rangle 2rr_0\langle 6\rangle + 2r\langle 2\rangle 2rr_0\langle 6\rangle] \\ &= -rr_0 6/(h^2+1)(w^2+2wh+r^2)[-n/2\sin(\theta-\theta_0)] \\ &\quad - 2nrr_0(w^2+2wh+r^2)\sin(\theta-\theta_0) \end{aligned}$$

$$= rr_0 n \sin(\theta-\theta_0)(h^2+1)^{-1}\{3(w^2+2wh+r^2)-2(h^2+1)(w^2+2wh+r^2)\}$$

$$= rr_0 n \sin(\theta-\theta_0)[-2+3/(h^2+1)](w^2+2wh+r^2). \quad (46)$$

In the above derivation, we have employed additional relations: $\Sigma\langle 5\rangle = 0$, $\Sigma\langle 2\rangle\langle 5\rangle = 0$, $\Sigma\langle 4\rangle\langle 5\rangle = \langle 4\rangle\Sigma\langle 5\rangle = 0$, $\Sigma\langle 2\rangle\langle 6\rangle = \langle 6\rangle\Sigma\langle 2\rangle = 0$, $\Sigma\langle 3\rangle\langle 5\rangle = -n/[2(h^2+1)]\sin(\theta-\theta_0)$; $\Sigma\langle 2\rangle\langle 3\rangle\langle 5\rangle = 0$, for proof see Appendices A and B.

(c) $\partial\Pi/\partial w = \Sigma 0.25EA1_{0i}^{-3}\varepsilon_{ti}\,\partial\varepsilon_{ti}/\partial w-\lambda$

$$= \Sigma 0.25EA(h^2+1)^{-1.5}[1+3r_0\langle 3\rangle][\langle 1\rangle-2r\langle 2\rangle+2rr_0\langle 4\rangle](2w+2h)-\lambda$$

$$= 2(h+w)\Sigma[\langle 1\rangle-2r\langle 2\rangle+2rr_0\langle 4\rangle+3r_0\langle 3\rangle\langle 1\rangle-3r_0\langle 3\rangle 2r\langle 2\rangle$$
$$+3r_0\langle 3\rangle 2rr_0\langle 4\rangle]-\lambda$$

$$= 2n(h+w)\{(w^2+2wh+r^2)+rr_0[-3/(h^2+1)+2]\cos(\theta-\theta_0)\}-\lambda \quad (47)$$

The r_0^2 terms have been neglected. Remember that $\Sigma\langle 2\rangle = \Sigma\langle 3\rangle = \Sigma\langle 1\rangle\langle 3\rangle = 0$, $\Sigma\langle 3\rangle\langle 4\rangle = \langle 4\rangle\Sigma\langle 3\rangle = 0$ and $\Sigma\langle 2\rangle\langle 3\rangle = n/[2(h^2+1)]\cos(\theta-\theta_0)$.

Obviously eqn (46) has two solutions:

solution 1: $\theta-\theta_0 = 0;\pi$

which, combined with eqn (41) and (43), will lead to:

$$2r(w^2+2wh+1+r^2)(h^2+1)-\pm r_0[6r^2+3(w^2+2wh+r^2)-(h^2+1)(6r^2+2w^2+4wh)] = 0 \quad (48)$$

$$2n(h+w)\{(w^2+2wh+r^2)\pm rr_0[-3/(h^2+1)+2]\}-\lambda = 0 \quad (49)$$

solution 2: $(w^2+2wh+r^2) = 0. \quad (50)$

From eqn (45) and (47), this will result in:

$$2r(h^2+1)-r_0[6r^2-(h^2+1)(6r^2-2h^2)]\cos(\theta-\theta_0) = 0 \quad (51)$$

$$2n(h+w)rr_0[-3/(h^2+1)+2]\cos(\theta-\theta_0)-\lambda = 0. \quad (52)$$

As a quick check, we make $r_0 = 0$, i.e. we have a perfect truss, it can be easily proved that the above two solutions coincide with the perfect solution (eqn 11). It is interesting to note that both solutions indicate a deformation pattern i.e r-w diagram, that does not depend on n-the number of trusses, because eqn (48) and eqn (50) do not contain n.

From eqn (48) and eqn (49), it is possible to solve λ and w in terms of r (see Appendix D), which has been visualised in Fig. 5 ($n = 4, h = 2, r_0 = 0.1$). Different from the load imperfection case (Fig. 3), we now seem to have an isolated solution, manifested by the loop on the left in Fig. 5a. A numerical simulation has also confirmed this, although the finite element analysis is quite tricky, where a restart with eigenmode injection (Wagner and Wriggers, 1988) is initiated at point A in Fig. 5a to jump to the isolated loop and many restarts are necessary to follow the whole path. So this is an interesting case where an isolated solution path exists. A further test on seven bar truss confirmed the fact that the deformation pattern is independent of the number of bars (Fig. 6a), though the load level is quite different (Fig. 6b and 6c)

Another difference from the load imperfection case is that we have two solutions instead of one. While the first solution lies in the plane decided by the θ_0, the second solution is three dimensional.

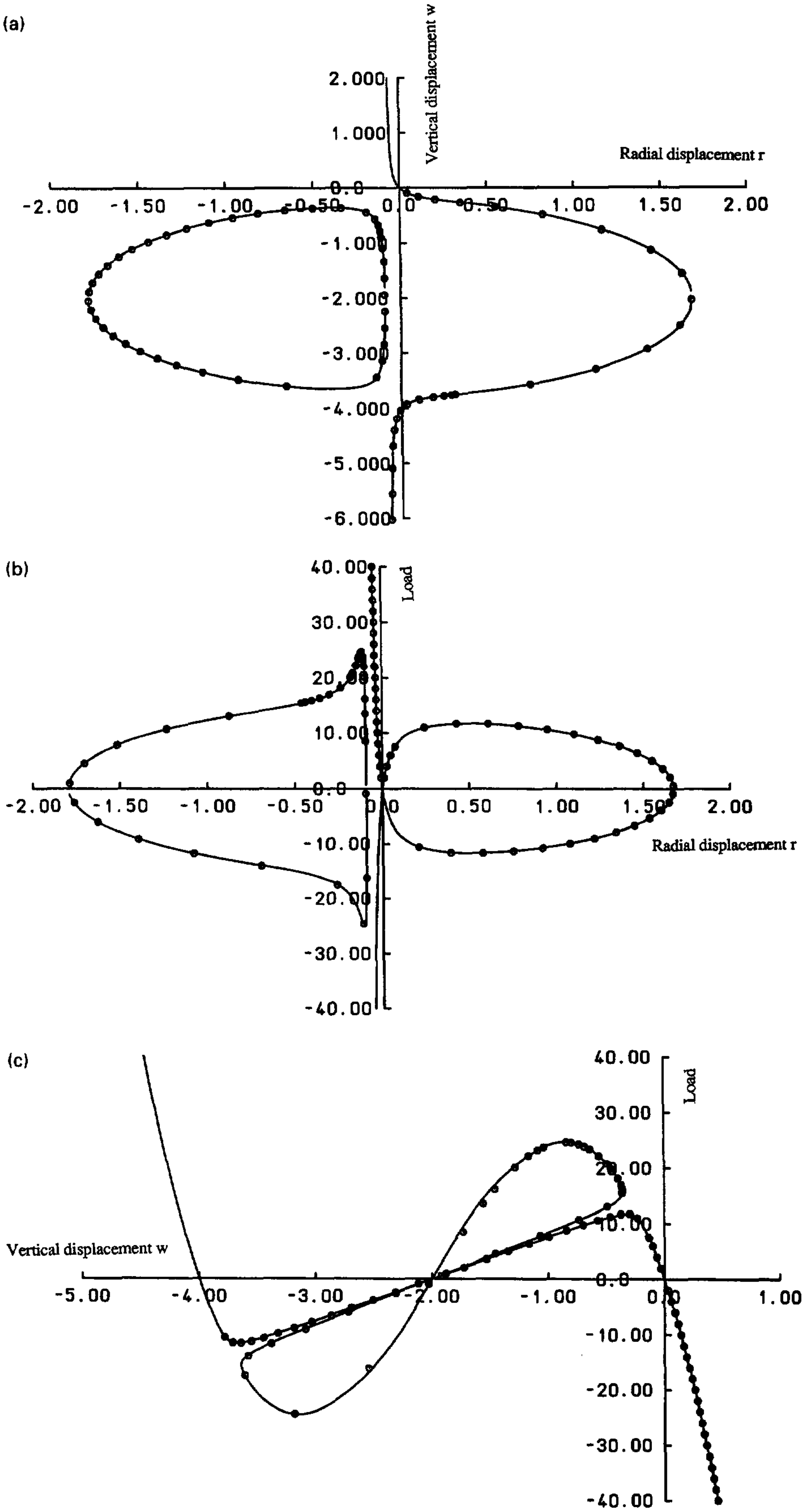

Fig. 5. The nonlinear response of a four bar truss with initial geometric imperfection: 'o' analytical; '——' finite element. (a) Vertical displacement w against radial dispalcement$r$; (b) load against radial displacement; (c) load against vertical displacement.

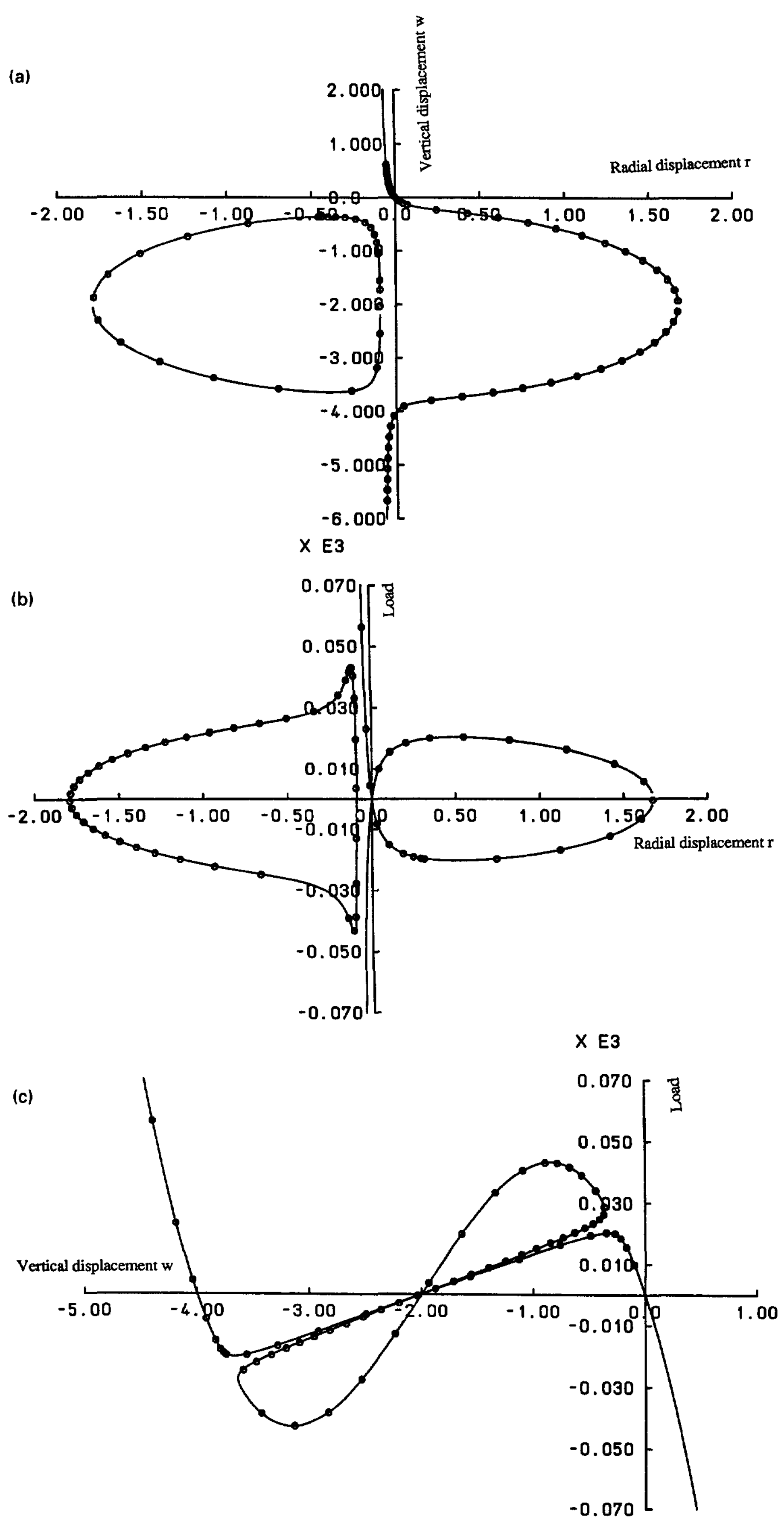

Fig. 6. The nonlinear response of a seven bar truss with initial geometric imperfection : 'o' analytical ; '——' finite element. (a) Vertical displacement w against radial displacement r ; (b) load against radial displacement ; (c) load against vertical displacement.

Great effort has been made to find the second solution given by eqns (50–52). The nonlinear finite element code employed failed to predict such a solution. In fact the existence of the second solution implies that the initial undeformed state is a bifurcation point, which is proved wrong by the positive definite stiffness matrix. This suggests that solution two might be created by the approximations employed in the analytical analysis.

6. SUMMARY AND FUTURE WORK

Close solutions for the nonlinear response of a simple truss have been obtained both with and without imperfection to illustrate some unique aspects of nonlinear system behaviour and to provide benchmark for nonlinear finite element solution algorithms. For the perfect truss, conditions of infinite bifurcation have been derived together with post-bifurcation solution. We have shown that finite element modelling of such a truss will always have a singular tangent stiffness matrix in the secondary path, which is a direct consequence of infinite bifurcation. This makes post-critical analysis numerically unstable. As for a perfect truss, we have found that unfolding leads to unconnected solutions, which demand special treatment in finite element simulation. In this paper, eigenmode injection is used.

We have so far restricted ourselves to static, elastic behaviour of the truss. A natural extension could be the dynamic and/or plastic response. Further we may replace bar by beam, which may introduce more intrinsic interaction into the system.

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APPENDIX A

(a) Prove that $\Sigma \cos\theta_j = 0$; $\Sigma \sin\theta_i = 0$; $\Sigma \cos 2\theta_j = 0$; $\Sigma \sin 2\theta_j = 0$.
Since

$$e^{i\theta j} = \cos\theta_j + i \sin\theta_j \tag{A1}$$

so

$$s = \Sigma\, e^{i\theta j} = \Sigma \cos\theta_j + i\Sigma \sin\theta_j \tag{A2}$$

which gives:

$$s = (1 - e^{in\theta 0})/(1 - e^{i\theta 0}) = (1 - e^{i2\pi})/(1 - e^{i\theta 0}) = 0 \tag{A3}$$

so

$$\Sigma \cos \theta_j = \Sigma \sin \theta_j = 0 \tag{A4}$$

where

$$\theta_0 = 2\pi/n.$$

Similarly we can prove that $\Sigma \cos 2\theta_j = 0$; $\Sigma \sin 2\theta_j = 0$.

(b) Prove that $\Sigma \cos^2 \theta_j = \Sigma \sin^2 \theta_j$

$$\Sigma \cos 2\theta_j = \Sigma(\cos^2 \theta_j - \sin^2 \theta_j) = \Sigma \cos^2 \theta_j - \Sigma \sin^2 \theta_j = 0. \tag{A5}$$

APPENDIX B

(a) $\Sigma \cos(k\theta_j + \theta) = 0$; $\Sigma \sin(k\theta_j + \theta) = 0$

where k is any integer, $\theta_j = j2\pi/n = j\theta_0$, where $j = 0, n-1$ and θ is any constant angle.

Since

$$e^{i(k\theta j+\theta)} = \cos(k\theta_j + \theta) + i \sin(k\theta_j + \theta) \tag{B1}$$

so

$$s = \Sigma\, e^{i(k\theta j+\theta)} = e^{i\theta} \Sigma\, e^{i(kj\theta 0)} = \Sigma \cos(k\theta_j + \theta) + i\Sigma \sin(k\theta_j + \theta) \tag{B2}$$

which gives:

$$s = e^{i\theta}(1 - e^{i(nk\theta 0)})/(1 - e^{i\theta 0}) = e^{i\theta 0}(1 - e^{i2k\pi})/(1 - e^{i\theta 0}) = 0 \tag{B3}$$

so

$$\Sigma \cos(k\theta_j + \theta) = \Sigma \sin(k\theta_j + \theta) = 0 \tag{B4}$$

(b)

$$\Sigma\langle 2\rangle\langle 2\rangle = \Sigma \cos(\theta_i - \theta)\cos(\theta_i - \theta) = \Sigma[\cos 2(\theta_i - \theta) - 1/2] = n/2; \tag{B5}$$

(c)

$$\begin{aligned}\Sigma\langle 2\rangle\langle 3\rangle &= \Sigma \cos(\theta_i - \theta)\cos(\theta_i - \theta_0)/(h^2+1) \\ &= 0.5/(h^2+1)\Sigma[\cos(2\theta_i - \theta - \theta_0) + \cos(\theta - \theta_0)] \\ &= 0.5n/(h^2+1)\cos(\theta - \theta_0);\end{aligned} \tag{B6}$$

(d)

$$\begin{aligned}\Sigma\langle 2\rangle\langle 2\rangle\langle 3\rangle &= \Sigma \cos(\theta_i - \theta)\cos(\theta_i - \theta)\cos(\theta_i - \theta_0)/(h^2+1) \\ &= 1/(h^2+1)\Sigma[\cos 2(\theta_i - \theta) - 1/2]\cos(\theta_i - \theta_0) \\ &= 0.25/(h^2+1)\Sigma[\cos(3\theta_i - 2\theta - \theta_0) + \cos(\theta_i - 2\theta + \theta_0)] - 0.5\Sigma \cos(\theta_i - \theta_0) = 0.\end{aligned} \tag{B7}$$

(e)

$$\begin{aligned}\Sigma\langle 3\rangle\langle 5\rangle &= \Sigma \cos(\theta_i - \theta_0)/(h^2+1)\sin(\theta_i - \theta) \\ &= 0.5/(h^2+1)\Sigma[\sin(2\theta_i - \theta - \theta_0) - \sin(\theta - \theta_0)] \\ &= -0.5n/(h^2+1)\sin(\theta - \theta_0)\end{aligned} \tag{B8}$$

(f)

$$\begin{aligned}\Sigma\langle 2\rangle\langle 3\rangle\langle 5\rangle &= \Sigma \cos(\theta_i - \theta)\cos(\theta_i - \theta_0)/(h^2+1)\sin(\theta_i - \theta) \\ &= 0.5/(h^2+1)\Sigma \cos(\theta_i - \theta_0)\sin 2(\theta_i - \theta) \\ &= 0.25/(h^2+1)\Sigma[\sin(3\theta_i - 2\theta - \theta_0) + \sin(\theta_i - \theta + \theta_0)] = 0.\end{aligned} \tag{B9}$$

APPENDIX C

Visualising eqn (31) and eqn (30)

eqn (30) divided eqn (31) gives:

$$\pm\kappa 2(h+w)[r^2 + 2wh + w^2] = 2r[r^2 + 2 + (h+w)^2 - l_0^2] \tag{C1}$$

which can be rearranged in terms of $(h+w)$ as:

$$\pm\kappa(h+w)^3 - r(h+w)^2 \pm \kappa(r^2 - h^2)(h+w) - r(r^2 + 1 - h^2) = 0. \tag{C2}$$

For a given r, we can find w from eqn (C2). Substituting r and w back into eqn (31), we can find λ. So we can define a range of r, say [0, 4] for $h = 2$, then increase it from 0 to 4 by certain number of increments, at which we also evaluate λ and w. Note that in the cylindrical co-ordinate system, r is always positive. However the $\pm$ sign in eqn (C2) corresponds to $\theta-\theta_0 = 0\,;\,\pi$, which are on the same line but in opposite direction. So in the physical space, $+$ relates to positive r, while $-$ relates to negative r, in the $\theta-\theta_0 = 0\,;\,\pi$ plane.

APPENDIX D

Visualising the first solution of geometrically imperfect truss- eqn (48) and eqn (49):

from eqn (48), we have:

$$2r[(w+h)^2-h^2+1+r^2](h^2+1)-\pm r_0\{6r^2+3(w+h)^2-3h^2+3r^2-(h^2+1)[6r^2+2(w+h)^2-2h^2]\} = 0 \tag{D1}$$

which can be rearranged as:

$$\{2r(h^2+1)-\pm r_0[3-2(h^2+1)]\}(w+h)^2 = -2r(-h^2+1+r^2)(h^2+1)\pm r_0[9r^2-3h^2-2(h^2+1)(3r^2-h^2)] \tag{D2}$$

or

$$[2(r\pm r_0)(h^2+1)-\pm 3r_0](w+h)^2 = \pm 3r_0(3r^2-h^2)+2(h^2+1)[-2r(-h^2+1+r^2)\pm r_0(3r^2-h^2)]. \tag{D3}$$

So for a given r, we can find w from the above equation. Substitute r and w into eqn (49), we can find λ. The plotting procedure can be the same as in Appendix C.

APPENDIX E

Visualising the second solution of the geometrically imperfect truss: eqn (50), eqn (51) and eqn (52):

start with r, we find w from eqn (50) and $\cos(\theta-\theta_0)$ from eqn (51), knowing all these, we may obtain λ from eqn (52).